The War through the Lens of Mathematics

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Abstract

This article discusses the application of mathematical models, specifically Lanchester's models in the context of warfare, to predict battle outcomes by considering factors such as the number of soldiers and their effectiveness. We review the use of these models for various historical battles and even in the context of contemporary conflicts, such as the Russian aggression in Ukraine but also in diverse areas, including animal behavior and strategy video games. These models provide valuable insights into battle dynamics, but they have limitations, particularly in accounting for the spatial arrangement of armies and the variability in an army's effectiveness based on morale and weapon types.

Keywords: Lanchester's Models, Warfare Mathematics, Battle Outcome Prediction, Military Strategy

La guerra a través de la lente de las matemáticas

Resumen

Este artículo analiza la aplicación de modelos matemáticos, específicamente los modelos de Lanchester en el contexto de la guerra, para predecir los resultados de la batalla considerando factores como el número de soldados y su efectividad. Revisamos el uso de estos modelos para diversas batallas históricas e incluso en el contexto de conflictos contemporáneos, como la agresión rusa en Ucrania, pero también en diversos ámbitos, incluido el comportamiento animal y los videojuegos de estrategia. Estos modelos proporcionan información valiosa sobre la dinámica de la batalla, pero tienen limitaciones, particularmente a la hora de tener en cuenta la disposición espacial de los ejércitos y la variabilidad en la efectividad de un ejército en función de la moral y los tipos de armas.

Palabras clave: modelos de Lanchester, matemáticas de guerra, predicción del resultado de la batalla, estrategia militar

doi: 10.18278/ijc.11.1.9

数学视角下的战争

摘要

本文探讨了数学模型的应用（特别是兰彻斯特模型在战争情境中的应用），通过考量士兵数量及其有效性等因素来预测战斗结果。我们述评了这些模型在不同历史战役中的使用，甚至在当代冲突情境下的使用（例如俄罗斯对乌克兰的侵略），以及在不同领域的使用，包括动物行为和策略视频游戏。这些模型为战斗动态提供了宝贵见解，但它们也有局限性，特别是在考虑军队空间布局以及基于士气和武器类型的军队效能的差异性这两方面。

关键词：兰彻斯特模型，战争数学，战斗结果预测，军事战略

In war, numbers alone provide no advantage. Sun Tzu (6th century BC - 5th century BC) in "The Art of War."

To determine the outcome of a battle, military strategists have long relied more on mathematics than crystal balls. Since each conflict has its own characteristics, seeking a single model for all confrontations is futile. However, it would be tedious to list all possible approaches. Let's focus on the early works on the subject to understand their utility and limitations.

There is a long tradition between mathematics and war. Archimedes is often presented as the first great scientist to have used his scientific knowledge to build war machines. During the siege of Syracuse in 212 BC, it is said that he built giant parabolic mirrors to ignite enemy sails by concentrating the sun's rays. Although the anecdote is certainly not true, it illustrates one of the first uses of science in warfare. On the other hand, the Greeks knew that war is better waged with mathematics as part of the train. Plato's "Socrates" explains that the commander needs arithmetic and geometry for displaying his troops optimally (Republica 525b). While World War II is very well known (Manhattan project, Enigma code, etc.) World War I was also an active period for mathematicians.

Frederick William Lanchester (October 23, 1868 - March 8, 1946) was a polymath who became interested in the influence of modern weapons on the outcome of conflicts as early as 1914 [4]. He considered that the method by which combat losses are computed is one of the most critical parts of any combat model. The Lanchester equations, which state that a unit's combat losses depend on the size of its opponent, are widely used for this purpose. In modern warfare, each weapon can eliminate multiple adversaries. To model the number of deaths in a battle, it's necessary to consider not only the number of soldiers but also firepower. This raises questions: Is it better to increase the number of soldiers or firepower?

What are the expected losses in a conflict? Can we estimate enemy losses? The first model by Lanchester (known as the linear model) is suited for ancient battles and has little practical significance.

The two models Lanchester (1916) conceived are called the linear law and the geometric law. The linear law is described as a series of one-on-one duels on the battlefield, most applicable to ancient warfare or area fire. In this scenario, the attrition ratio of the two forces is independent of the force ratio. The geometric law, associated with modern warfare, describes combat where multiple units of a force can focus their aimed fire onto single targets. In this concentrated fire model, the attrition that a force suffers is proportional to the number of enemies.

Lanchester's geometric model was initially developed to simulate aerial combat (a major innovation during World War I) and focuses on the number of surviving soldiers in a battle. He also extends the theory to include heterogeneous force composition, naval warfare, and other battle types.

Lanchester's geometric model was initially developed to simulate aerial combat (a major innovation during World War I) and focuses on the number of surviving soldiers in a battle. Even if it is fundamental to know the number of soldiers at a given time, it is crucial to consider the rate of change. The objective of Lanchester's model is to calculate the evolution of the number of soldiers over time and, ultimately, the number of surviving soldiers. From a mathematical perspective, the concept of a derivative with respect to time begins to emerge, i.e., the rate of disappearance of an army as a function of time. As the number of soldiers decreases, this derivative must be negative. Furthermore, this decrease depends on the effectiveness of the opposing army. In the end, if we consider two armies A and B, and denote A(t) and B(t) as the number of soldiers at time "t," and α and β as the effectiveness of armies A and B, we obtain a system of differential equations where the derivative with respect to time of A(t) is the product of beta and B(t) (with a negative sign); similarly, for the derivative of B(t). (See image 1).

$$\frac{\delta A(t)}{\delta t} = -\beta B(t)$$

$$\frac{\delta B(t)}{\delta t} = -\alpha A(t)$$

Image 1. System of differential equation from Lanchester's model

Solving this system of equations is relatively straightforward. In the conventional approach, A(0) and B(0) represent the initial strengths of the two armies at the beginning of the battle. For simplicity, let's assume that the battle ends due to a lack of combatants. If camp A wins the battle at time T, then B(T) = 0, and it can be demonstrated that the number of survivors is:

$$A(T) = \sqrt{A^2(0) - \frac{\beta}{\alpha} B^2(0)}.$$

If camp B wins the battle at time T, A(T) = 0 and

$$\sqrt{B^2(0) - \frac{\alpha}{\beta} A^2(0)}$$

Before discussing the applications and limitations of the model, let's see what this model teaches us. First, it's evident that the influence of efficiency is linear, while the number of soldiers at the beginning of the conflict has a quadratic influence. The statement that quantity is more important than quality probably stems from this criterion. The Lanchester square model is a continuous model of time and state in practice, often regarded as a kind of mean-value model. This implies that it's better to be more than twice as numerous as to be more than twice as efficient. The "3:1 rule" in ground combat stipulates that an attacking force should have a 3-to-1 advantage over a defending force to succeed. This rule originated from operations research. Let note that the logic behind this model deteriorates as the number of operational units decreases, and it certainly collapses when the number of operational units is reduced to zero.

This model describes combat between two homogeneous forces using long-range weapons such as tanks, revolvers, and machine guns. Both fight under the assumption of complete tactical information. Complete tactical information means that an arbitrary operational unit is always capable of detecting at least one of the many hostile operational units because it can kill. In addition, it is assumed that all the operational units on each side can share information fully and coordinate their firepower among the hostile operational units. In fact, firepower is the only limiting factor. This assumption may be completely unrealistic, as it is difficult to identify many modern combat situations where the assumption is respected by both sides. For example, very few army operations are carried out with complete tactical information on both sides. It also doesn't account for the spatial arrangement of armies, and the effectiveness of an army varies based on morale and types of weapons. Furthermore, the deterministic nature of the model means that the same conditions will always yield the same result. Current models prefer simulations that provide probabilities of victory.

Economic and military growth require investment. The more you invest, the faster you grow. The more the state invests in the military, the less it can invest in the economy. Military expenditure has the opportunity cost of reducing economic expenditure. Based on these assumptions, some authors proposed to extend Lanchester model with the growth of a nation's resources as a function of the amount of its resources invested in economic growth as well as the amount it invests in military preparations for combat.

In Lanchester's model both sides apply the same type of tactics and firing techniques; the battles are symmetric. Asymmetric engagements occur when the two sides apply different tactics. One such asymmetric combat situation occurs when regular forces of a state fight guerrillas or insurgents who apply irregular warfare tactics. A different manifestation of asymmetry in Lanchester models is when the two sides employ profoundly different tactics. Consider an aimed-fire situation where a homogeneous Blue force is engaged in battle with a heterogeneous Red force comprising n units that are different in terms of fire-effectiveness and vulnerability. Lin and MacKay showed that the optimal tactic for Blue is such that Blue should not spread out its effort but rather concentrate all its fire on one Red adversary at a time. At any given time, Blue should concentrate its fire on the adversary for which the "product" of its vulnerability and threat is the highest.

The asymmetric models apply to irregular warfare where well-organized, military forces of the state confront low-signature guerrilla fighters. These models focus on the asymmetry in information and its impact on battlefield outcome. Another crucial component in irregular-warfare scenarios is the civilian population who, on the one hand, are subject to violent actions by the guerrillas, and on the other hand, may be a source of support and provider of hiding places for the guerrillas' fighters. This question was studied by Kress in [9].

Lanchester's equations essentially model the attrition between two opposing forces. They capture a duel, force-on-force, situation. However, recent, as well as some historical, conflicts involve more than two opposing forces. The Bosnian Civil War (Croatia, Bosnia Herzegovina, Serbia, NATO), the Iraq Civil War (Coalition Forces, Sunni Militia, Shia Militia), and most recently, the war in Syria (Assad Regime Forces, Free Syrian Army, Hezbollah, Kurds, Russia, Turkey) are just a few examples of such multilateral violent conflicts. Two recent papers extend the classical Lanchester theory to the case where the attritional conflict comprises more than two players. It is important to note a profound difference between two- and multiple-player Lanchester models. In a two-player (force-on-force) conflict, the Lanchester models are purely descriptive; they simply capture the attrition on both sides as a function of the initial strengths and the attrition rates of the two players: Blue and Red. No decision is required, by either player, during the engagement. However, in a multiple-player conflict, each player has to decide how to allocate its strength among the other adversaries so as to maximize its own chances to be the victor. This decision, common to all other players, leads to a prescriptive model where each one of n players ($n > 2$) has to dynamically allocate its existing strength among its $n - 1$ adversaries.

Kress et al. [8] studied this question as a differential game where each player wishes to maximize its own surviving force minus that of its enemies. The outcome of the analysis is surprising: either a player is strong enough to win over the other players combined in a coalition against itself, or all players are locked in a stalemate that leads to their mutual demise. In the case of three players, this conclusion stands in contrast to sequential-engagement scenarios in which the weakest player can achieve an advantage.

Lastly, for those familiar with differential equation systems, they might have recognized in Lanchester's model a simplified version of the Lotka-Volterra equations. This model, called the predator-prey model, deals with the interaction between two species, one being a predator and the other its prey. It was introduced a decade after Lanchester's work.

In Lotka-Voltera's model, prey are assumed to have an unlimited source of food and to reproduce exponentially if they are not subject to predation. The rate of predation on prey is assumed to be proportional to the frequency of encounters between predators and prey. The variation in the number of preys is given by its own growth minus the predation rate applied to it. Similarly, the variation in the predator population is given by the growth of this population, minus the number of natural deaths. Predators thrive when prey is plentiful, but eventually exhaust their resources and decline. When the predator population has declined sufficiently, the prey that has benefited from the respite reproduce and their population increases again. This dynamic continues in a cycle of growth and decline. This model was used for guerrilla [10]. The growth term reflects guerrilla recruitment, which depends on the interaction between guerrilla forces and the population they control while predation reflects losses of guerrillas by either death or capture or defection to the regulars. At the same time, the regular recruitment rate is the rate of increase of the regulars if there were no guerrillas and the predation is losses to guerrillas, depending on the interaction of the guerrillas with the regulars.

Despite all its limitations, this model has been successfully applied to various situations [2]. Lanchester's laws have been used to explain the characteristics of battles among a variety of animals that fight in groups, including termites, ants, birds, and chimpanzees [3]. Among the proposed implications, one can mention investment in offspring, decision rules on when to engage in combat, and differences in size between native and non-native species. Today, it is also widely used in strategy video games whenever there are battles, such as in Age of Empires, for instance, or to model the outcome of a game like StarCraft [6].

Lanchester's model has been applied to various historical battles, including the Battle of Trafalgar, the Battle of Kursk in 1943 involving German and Soviet tanks, the Battle of Berlin, the Battle of the Bulge during World War II, and the naval conflict in the Atlantic during which German U-Boats inflicted damage on

Allied convoys. In a 1954 article, Engel applied Lanchester's model to the Battle of Iwo Jima.

The battle was towards the end of the war and the Pacific Theater campaign. The data in this paper covers the force attrition from the 19th of February through the 26th of March 1945. Japan was on the defensive at the strategic as well as the operational levels. The island was used by the Japanese military throughout the war as a waypoint and relay for communications, aircraft, and supplies between the Japan mainland and the rest of the Southwest Pacific.

The source of the data is the daily casualty recordings from the historical accounts and official records from the battle of Iwo Jima. This information was compiled during the operation by the U.S. Marine Corps Historical Division. The American army would lose around 26,000 soldiers (including 7,000 deaths). Such detailed casualty data for the Japanese forces is not available, however, and the only usable data regarding Japanese force size over time is that there were 21,500 at the start of the battle (D+0) and approximately zero at the end of the recorded fighting (D+35). D+28 is the official end of the battle as declared by the operational U.S. Marine command at the time; and, although there were some residual Japanese forces deep within buried tunnels and bunkers that were exposed over the following days, weeks, months, and even years, they were relatively low in number and are generally not considered for the analysis of the data.

This battle aligns with many of the assumptions of Lanchester's model since the island was isolated, and the Japanese defenders were nearly all killed. The island was attacked by approximately 73,000 American soldiers (54,000 on the first day, 6,000 on the second, and 13,000 on the sixth day). It's relatively easy to reconstruct the dynamics of the model and estimate the parameters. The Japanese efficiency was 0.0577, while the American efficiency was 0.0106. Therefore, there was an efficiency ratio of 5.4 between the two armies. It's also possible to reconstruct the dynamics of the battle and explore what would have happened with different scenarios. For example, if the Americans hadn't sent reinforcements on the second and sixth days, there would likely have been an additional 7,000 casualties, and the battle would have lasted 66 days instead of 36. That's a one-month difference. Conversely, if all troops had been landed on the first day, it would have saved 2,000 lives (and one day of battle). Finally, it would have taken at least 31,300 Japanese soldiers to withstand the American assault.

Today the current European military landscape is heavily influenced by the Russian aggression in Ukraine, and we can already see the application of Lanchester's model to this conflict [5]. The balance of power in the War in Ukraine is very similar to that in the American Civil War. In 2022 Russia had about 4:1 advantage in population. Based on these numbers, Ukraine has about as much chance of winning the conflict as the Confederacy did. But this prediction comes from a very simple model, and reality can deviate from it due to several factors (such as morale, munition production, skills, logistics, etc.). Depending on how important these factors are, the prediction could be substantially different. In other words, we have alternative predictions resulting from different assumptions—finding out which of these alternatives matches the data best is what the scientific method is about. Math gives a model, but life can be more complicated.

Although addressing the challenges posed by heavy artillery in terms of target location and trajectory calculations demanded expertise in mathematics, World War I revealed numerous technological and military advancements that also hinged on meticulous mathematical analyses. Whether it was keeping an airplane airborne or facilitating its descent, various mathematical problems needed solutions. Additionally, the advancement and refinement of technologies like sonar or wireless telegraphy, along with the practical military application of cartography and meteorology on a daily basis, demanded mathematical skills that, while not necessarily groundbreaking, were nonetheless scarce among the enlisted personnel. On the other hand, human interactions supply many phenomena, which can be modelled and analyzed with applied mathematics. Lanchester's model is one of the most famous models for the role of the military strategy (and the linked decision problems) during a conflict between two or more armies. Even if hardware of weapons systems is fundamental, the behavioral aspects of combat are also very important and historical data offer rich opportunities for studying the effects of tactics.

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